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CONTRACT No [illegible](23)
PROJECT No. NR 064-453

ABLATION OF A SOLID SPHERE OF A LOW CONDUCTIVITY MATERIAL

by

Richard F. Parisse

POLYTECHNIC INSTITUTE OF BROOKLYN

DEPARTMENT
of
AEROSPACE ENGINEERING
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APRIL 1964

FINAL REPORT NO. [illegible]

Contract No. Nonr 839(23)
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PIBAL Report No. 683

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ABSTRACT

An approximate solution is presented for determining transient temperature distributions and ablation rates for solid spheres of low thermal conductivity and constant thermal properties, subjected to point symmetric aerodynamic heat inputs. A short time solution for the temperature which is valid prior to ablation, is developed by use of Laplace transforms. The ablation solution is derived by approximation of the radial temperature profile by an exponential function and utilization of the heat balance technique. This results in a second-order, nonlinear, ordinary differential equation for the position of the ablating surface.

Results of the numerical integration of the approximate solutions are compared with experimental data obtained at the stagnation point of a Plexiglas hemisphere tested under hypersonic flow conditions. A discussion of the techniques employed, as well as the comparison with experimental data, appear in the body of this report.

SYMBOLS

a	initial radius of sphere
c	specific heat
h	heat transfer coefficient
k	thermal conductivity
L	heat of sublimation or melting
P_s	stagnation pressure
Q_o	constant heat flux
r	radial coordinate
r_s	radius of ablating heated surface
r'	nondimensional radial coordinate, r/a
t	time
t_m	melt time
t'	nondimensional time, $\kappa t/a^2$
T	temperature
T_a	heated surface temperature
T_i	initial temperature
T_m	melt or sublimation temperature
T_o	radial temperature profile from pre-melt analysis
T_s	stagnation temperature of flow field
$\bar{T}$	nondimensional temperature, $(T - T_i)/(T_s - T_i)$
u	transformed variable, rT or $r(T_s - T)$
$\bar{u}$	Laplace variable

γ	nondimensional heat transfer parameter, $aQ_o/\rho L\kappa$
κ	thermal diffusivity
η	nondimensional ablation radius, r_s/a
φ	nondimensional material physical parameter, $\dfrac{k(T_m - T_i)}{\rho L\kappa}$
ρ	density of solid
τ	nondimensional time, $\dfrac{\kappa(t - t_m)}{a^2}$
μ	nondimensional heat transfer parameter, $\dfrac{ah - k}{k}$
$(\dot{\ })$	indicates differentiation with respect to τ

INTRODUCTION

The high inputs and aerodynamic forces encountered by a space vehicle upon re-entry into the atmosphere have created numerous areas of interest for the scientist and engineer. One area of recent emphasis has been ablation, or, the utilization of the latent heat of vaporization or sublimation of the material in order to reduce the heat input to the vehicle structure.

Considerable research has been performed in problems of transient temperature distributions in bodies undergoing phase changes. References [1] through [7], together with the lists of references therein, contain a fairly comprehensive bibliography of this subject.

The current research was motivated by the desire to obtain a prediction of ablation rates and temperature profiles for thick-walled hemispheres of low thermal conductivity subjected to aerodynamic heat inputs. Tests of Plexiglas and Nylon hemispherical models are now being conducted in the hypersonic tunnel of the Polytechnic Institute of Brooklyn. The experimental results available on one of the models are compared to the present theory. A more extensive comparison will be the subject of a future report.

In this report an approximate theoretical method is developed for determining transient temperature distributions, together with ablation rates for solid spheres of low thermal conductivity and constant thermal properties subjected to point symmetric aerodynamic heat inputs. Since the high thermal input conditions encountered in re-entry produce steep radial temperature gradients at the heated surface of a material of low thermal conductivity, the

material temperature at a short distance from the heated surface is essentially the same as the initial material temperature. Consequently, the solution for a spherical shell of sufficient thickness may be approximated by the solution for a solid sphere. The problem as developed in this report is divided into two parts: (1) the pre-melt solution, and (2) the ablation solution.

The pre-melt solution utilizes Laplace transformations which result in a short-time solution to the heat conduction equation [8]. This solution is considered valid until the sublimation or melting temperature of the body is reached on the heated surface. The determination of the time at which the above condition occurs is of primary importance in correlating experimental ablation data with the ablation theory.

The ablation solution is approximated by employing the Goodman heat balance technique [2]. An exponential temperature profile is assumed in the unablated sphere and substituted into the heat conduction equation. The resulting equation is integrated over the region of the solid, and yields an ordinary, nonlinear, second-order, differential equation relating the ablation depth to time. The latter equation is solved by numerical techniques.

The ablation solution, used together with the pre-melt solution, provides a simple and reasonable approximation for the ablation of low conductivity spherical shells.

PRE-MELT SOLUTION

A short-time solution for the temperature distribution in a solid sphere subjected to a point symmetric aerodynamic heat input is obtained by utilizing Laplace transformations. The appropriate form of the heat conduction equation is

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) \quad ; \qquad 0 < r < a \tag{1}$$

with the conditions

$$h(T_s - T_a) = k(\partial T/\partial r)_{r = a} \tag{2a}$$

$$T(r,0) = T_i \tag{2b}$$

$$T(0,t) \text{ is finite} \tag{2c}$$

where the sumbols are defined as

a	initial radius of sphere
h	heat transfer coefficient
k	thermal conductivity
r	radial coordinate
t	time

T	temperature
T_a	heated surface temperature
T_i	initial temperature of material
T_s	stagnation temperature of flow field
κ	thermal diffusivity

Equations (1) can be transformed into a more convenient form by the substitution

$$u = r(T_s - T)$$

The result is

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial r^2} \quad ; \qquad 0 < r < a \tag{3}$$

and equations (2) become

$$\frac{\partial u}{\partial r} + \frac{\mu}{a} u = 0 \; ; \qquad \text{at } r = a \tag{4a}$$

$$u(r,0) = r(T_s - T_i) \tag{4b}$$

$$u(0,t) = 0 \tag{4c}$$

where $\mu = (ah - k)/k$.

The Laplace transformation is now applied to Eq. (3)

$$\int_0^\infty e^{-pt} \frac{\partial^2 u}{\partial r^2}\, dt - \frac{1}{\varkappa} \int_0^\infty e^{-pt} \frac{\partial u}{\partial t}\, dt = 0$$

and by using Eq. (4b), the resulting subsidiary equation is

$$\frac{d^2\bar{u}}{dr^2} - q^2\bar{u} = - \frac{(T_s - T_i)r}{\varkappa} \quad , \qquad (5)$$

where

$$q^2 = p/\varkappa \ .$$

Equations (4a) and (4c) become

$$\frac{d\bar{u}}{dr} + \frac{\mu}{a}\bar{u} = 0 \quad , \qquad \text{at } r = a \qquad (6a)$$

$$\bar{u} = 0 \quad , \qquad \text{at } r = 0 \ . \qquad (6b)$$

A general solution to Eq. (5) is

$$\bar{u} = C_1 e^{qr} + C_2 e^{-qr} + \frac{(T_s - T_i)r}{q^2 \varkappa}$$

where C_1 and C_2 are constants. These constants are evaluated by applying

Eqs. (6a) and (6b). The result in series form is as follows:

$$\bar{u} = \frac{(T_s - T_i)r}{p} - \frac{(1+\mu)(T_s - T_i)}{p} \sum_{n=0}^{\infty} (-1)^n \frac{(q-\mu/a)^n}{(q+\mu/a)^{n+1}} \left\{ e^{-q[(2n+1)a-r]} - e^{-q[(2n+1)a+r]} \right\} \tag{7}$$

For a short-time solution, Eq. (7) may be truncated after the n = 0 term. For the range of parameters considered in this report the higher order terms of Eq. (7) can be shown to be negligible.

The inverse transformation of Eq. (7) for n = 0 is

$$\begin{aligned} \bar{T} = \frac{1}{r'}\left[\frac{1+\mu}{\mu}\right] \Bigg\{ & \mathrm{erfc}\, \frac{1 - r'}{2\sqrt{t'}} - \mathrm{erfc}\, \frac{1 + r'}{2\sqrt{t'}} \\ & - \exp\left[\mu(1-r') + \mu^2 t'\right] \mathrm{erfc}\left[\frac{1 - r'}{2\sqrt{t'}} + \mu\sqrt{t'}\right] \\ & + \exp\left[\mu(1 + r') + \mu^2 t'\right] \mathrm{erfc}\left[\frac{1 + r'}{2\sqrt{t'}} + \mu\sqrt{t'}\right]\Bigg\} \end{aligned} \tag{8}$$

where

$$t' = \frac{\kappa t}{a^2}$$

$$r' = r/a$$

$$\bar{T} = \frac{T - T_i}{T_s - T_i} \quad .$$

Equation (8) represents the short-time solution for the temperature-time history at any point on or within the sphere.

In order to correlate the experimental ablation data with the ablation theory, the time (t_m) at which the melting temperature (T_m) of the material is initially reached on the heated surface must be determined. Surface temperatures versus time for various values of the heat input parameter (μ) have been calculated, using Eq. (8), and the results are shown in Fig. 1. For a given material and heat input condition Fig. 1 can be used to determine the melting time (t_m).

Once t_m has been determined, the radial temperature profile at $t = t_m$ can be calculated from Eq. (8). The determination of this profile is not essential for the solution of the ablation problem, but it must be shown here as a justification for the simplified profile chosen for the ablation theory. One such profile is presented in Fig. 2 as a representative case.

ABLATION SOLUTION

The transient temperature distributions and ablation rates for the solid sphere are approximated by employing the heat balance technique due to Goodman [2]. The solution is considered valid from the time melting commences on the heated surface to the time that the predicted temperature increase at the center of the sphere ceases to be a negligible percentage of $(T_m - T_i)$. The initial assumptions made in this analysis are as follows:

a. the heated surface remains at the melt or sublimation temperature (T_m).

b. the melt or products of sublimation are immediately removed upon formation.

c. the thermal properties of the solid are independent of temperature.

d. the heat flux (Q_o) remains constant[1].

The appropriate form of the heat conduction equation is

$$\frac{\partial T}{\partial t} = \kappa\left[\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right] ; \quad \begin{array}{l} 0 \leq r \leq r_s(t) \\ t \geq t_m \end{array} \qquad (9)$$

The function $r_s(t)$ denotes the time-dependent radial coordinate of the receding outer surface of the sphere.

[1] In the case of aerodynamic heating this assumption, together with the assumption that the heated surface remains at a constant temperature, is analogous to assuming that the heat transfer coefficient (h) is constant.

The boundary and initial conditions to be satisfied are

$$T(r_s,t) = T_m \tag{10a}$$

$$T(r,t_m) = T_o(r,t_m) \tag{10b}$$

$$T(0,t) \text{ is finite} \tag{10c}$$

$$r_s(t_m) = a \tag{10d}$$

$$Q_o = k \left(\frac{\partial T}{\partial r}\right)_{r_s,t} - \rho L \frac{dr_s}{dt} \tag{10e}$$

where $T_o(r,t_m)$ is the temperature profile at time (t_m) determined from the pre-melt analysis, and ρ and L are the density and heat of sublimation of the material, respectively.

Equation (9) is converted to a more convenient form by applying the transformation

$$u = rT$$

whereupon

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial r^2}, \qquad 0 \leq r \leq r_s,\ t \geq t_m \quad . \tag{11}$$

Integrating Eq. (11) over the volume of the solid,

$$\int_0^{r_s} \frac{\partial u}{\partial t}\, dr = \varkappa \int_0^{r_s} \frac{\partial^2 u}{\partial r^2}\, dr \quad ,$$

or

$$\frac{d}{dt}\int_0^{r_s} u dr - u(r_s,t)\,\frac{dr_s}{dt} = \varkappa \left[\left(\frac{\partial u}{\partial r}\right)_{r_s,t} - \left(\frac{\partial u}{\partial r}\right)_{o,t} \right] . \qquad (12)$$

Reintroducing Tr for u, Eq. (12) may be expressed in the form

$$\frac{d}{dt}\int_0^{r_s} Tr dr - r_s T(r_s,t)\,\frac{dr_s}{dt} = \varkappa \left[r_s \left(\frac{\partial T}{\partial r}\right)_{r_s,t} + T(r_s,t) - T(0,t) \right] \qquad (13)$$

Condition (10b) is now satisfied approximately by assuming the temperature profile

$$\frac{T - T_i}{T_m - T_i} = \exp\left\{ -\left[\frac{Q_o + \rho L(dr_s/dt)}{k(T_m - T_i)} \right](r_s - r) \right\} \qquad (14)$$

This temperature profile also satisfies conditions (10a), (10c) and (10e).

Substitution of (14) into (13) yields the following equation:

$$\left[2(1 - e^{-\alpha}) - \alpha(1 + e^{-\alpha})\right] \frac{d^2 r_s}{dt^2}$$

$$= \frac{k}{\rho L}\left(\frac{\alpha}{r_s}\right)^3 (T_m - T_i)\left\{ \varkappa\left[\alpha + (1 - e^{-\alpha})\right] + \left[\alpha - (1 - e^{-\alpha})\right]\left(\frac{r_s}{\alpha}\right)\frac{dr_s}{dt} \right\} \qquad (15)$$

where

$$\alpha = \frac{[Q_o + \rho L(dr_s/dt)]r_s}{k(T_m - T_i)}$$

It is now assumed that α is sufficiently large, such that $e^{-\alpha} \ll 1$. Neglecting the exponential terms compared to unity in Eq. (15), the resulting equation is:

$$\frac{k}{\rho L} \frac{(T_m - T_i)}{(2 - \alpha)} \left(\frac{\alpha}{r_s}\right)^3 \left[\varkappa(\alpha+1) + \left(\frac{r_s}{\alpha}\right)(\alpha - 1) \frac{dr_s}{dt}\right] = \frac{d^2 r_s}{dt^2} . \qquad (16)$$

Equation (16) is non-dimensionalized by the transformation

$$\tau = \frac{\varkappa(t - t_m)}{a^2}$$

$$\eta(\tau) = r_s/a .$$

The result is

$$\ddot{\eta} = - \frac{(\gamma + \dot{\eta})^2}{\varphi^2[\eta(\gamma+\dot{\eta}) - 2\varphi]} \left\{\varphi\left[(\gamma + \dot{\eta}) - \varphi\dot{\eta}\right] + \eta(\gamma + \dot{\eta})\left[\gamma + (1 + \varphi)\dot{\eta}\right]\right\} \qquad (17)$$

where

$$\gamma = \frac{aQ_o}{\rho L\varkappa}$$

$$\varphi = \frac{k(T_m - T_i)}{\rho L\varkappa}$$

$$\dot{\eta} = \frac{d\eta}{d\tau}$$

$$\ddot{\eta} = \frac{d^2\eta}{d\tau^2}$$

A stepwise numerical integration procedure is used for the solution of Eq. (17), starting with the initial conditions on η and $\dot{\eta}$. It can be shown from condition (10d) that η at $\tau=0$ is unity, and for continuity of the heat flux at the so-called melt time, that $\dot{\eta}$ at $\tau = 0$ is zero. These initial values of η and $\dot{\eta}$ are now used to find $\ddot{\eta}$ (0) from Eq. (17). A value of the time interval ($\Delta\tau = \tau_2 - \tau_1$) is chosen, and the following Taylor series expansions are used to proceed:

$$\begin{aligned} \eta(\tau_2) &= \eta(\tau_1) + \dot{\eta}(\tau_1)\Delta\tau + \ddot{\eta}(\tau_1)\frac{(\Delta\tau)^2}{2!} \\ \dot{\eta}(\tau_2) &= \dot{\eta}(\tau_1) + \ddot{\eta}(\tau_1)\Delta\tau \end{aligned} \tag{18}$$

The values of $\eta(\tau_2)$ and $\dot{\eta}(\eta_2)$ thereby obtained are substituted into Eq. (17) to obtain $\ddot{\eta}(\tau_2)$. This process is then repeated for each successive time interval.

The resulting time history of the ablating surface (η vs τ) for various values of the heat input parameter (γ), and two values of the

material parameter (φ), are shown in Figs. 3 and 4. The corresponding time histories of the ablation rates ($\dot{\eta}$ vs τ) are shown in Figs. 5 and 6. The values of φ used correspond to the physical properties of Plexiglas ($\varphi = 0.511$) and Nylon ($\varphi = 0.108$).

The values of the ablation rate and ablation radius so determined can be substituted into Eq. (14) to yield the temperature-time history at any point within the body.

COMPARISON WITH EXPERIMENT

The theoretical results of the preceding sections are now compared with an experimental result obtained at the stagnation point of a Plexiglas hemisphere tested in the hypersonic tunnel of the Polytechnic Institute of Brooklyn. The shroud technique was utilized to produce pressure distributions and heat transfer rates corresponding to high speed re-entry of the hemisphere.

Dimensions of Model	
Outer Radius, a	3.875 in
Inner Radius, b	2.875 in
Test Conditions	
Stagnation Pressure, P_s	19.5 psia
Stagnation Temperature, T_s	$1700^\circ R$
Heat Transfer Coefficient at Stagnation Point, h	$0.0155 \frac{BTU}{Ft^2 sec ^\circ R}$
Initial Temperature of Model, T_i	$540^\circ R$
Duration of Test Run, t_f	37 sec
Ablation at Stagnation Point, $a - r_s$	0.053 in
Assumed Physical Properties	
Sublimation Temperature, T_m	$1200^\circ R$
Diffusivity, κ	1.67×10^{-4} in^2/sec
Conductivity, k	$3.61 \times 10^{-4} \frac{BTU\ in}{Ft^2 sec ^\circ R}$
Density, ρ	74.4 lb/ft^3
Specific Heat, c	0.35 BTU/lb$^\circ$R
Heat of Sublimation, L	450 BTU/lb

Pre-Melt Solution

From the above data the parameters necessary to use Fig. 2 can be calculated. These are

$$\mu = \frac{ah - k}{k} = 165$$

and

$$\bar{T}_m = \frac{T_m - T_i}{T_s - T_i} = 0.569 \ .$$

From Fig. 2, $t'_m = 3.30 \times 10^{-5}$,

and hence

$$t_m = \frac{a^2 t'_m}{\kappa} = 2.96 \text{ seconds}$$

Albation Solution

The ablation parameters, φ and γ, calculated from the given data are

$$\varphi = 0.511$$

$$\gamma = 62.8$$

The non-dimensional time corresponding to the conclusion of the test (τ_f) can be determined from the given data and from the pre-melt solution. That is

$$\tau_f = \frac{\varkappa(t_f - t_m)}{a^2} = 3.78 \times 10^{-4}$$

The total ablation depth predicted by the theory can now be determined by using Fig. 3. The result is

$$a - r_s = 0.047 \text{ in.}$$

The actual ablation depth measured at the stagnation point is 0.053 in. and hence the theoretical result is within 12% of the experimental result.

s

ACKNOWLEDGEMENT

The author gratefully acknowledges the criticisms and suggestions of Dr. S.V. Nardo and Of Dr. T.J. Lardner.

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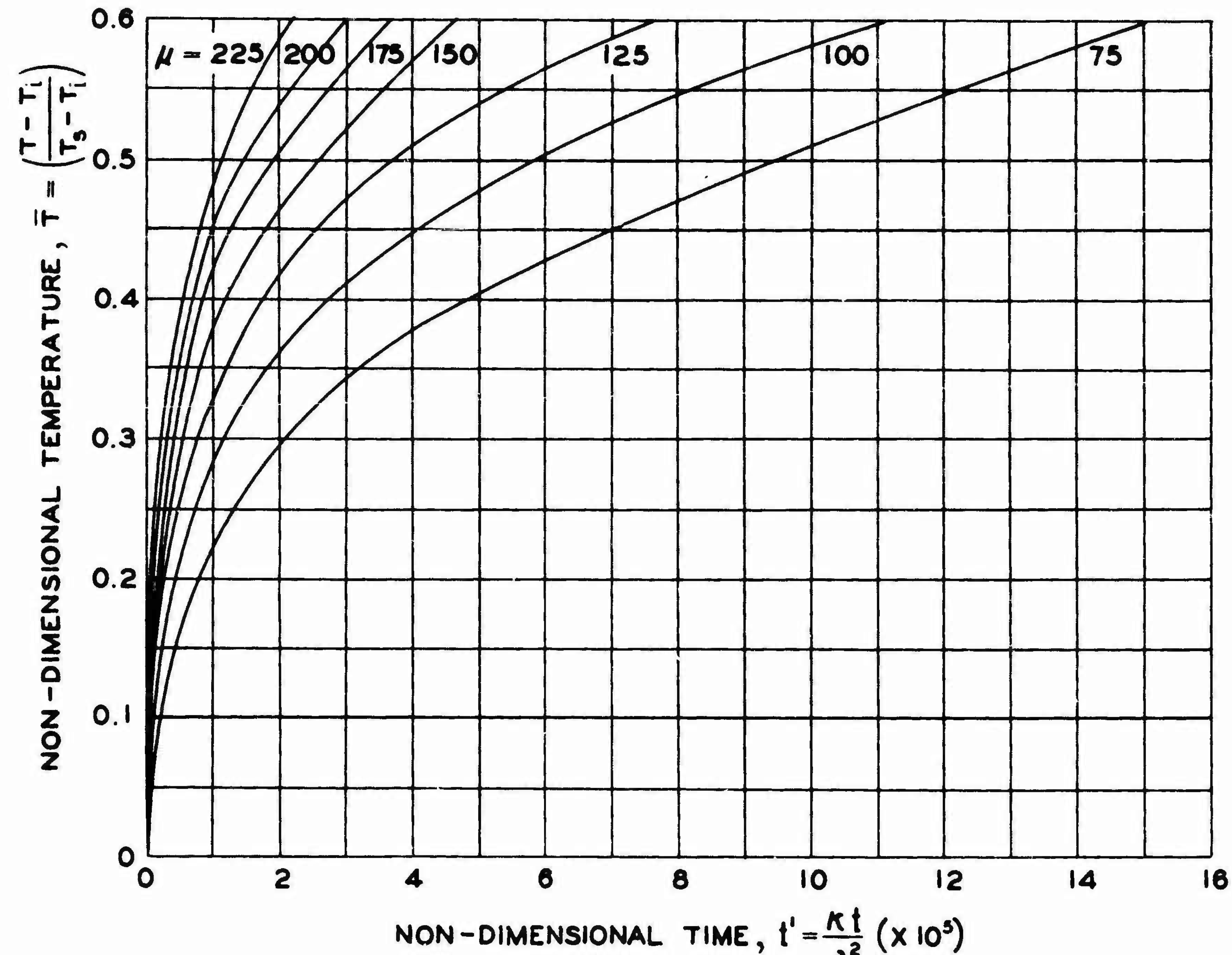

FIG. 1
SURFACE TEMPERATURE VS TIME FOR VARIOUS HEAT INPUT PARAMETERS

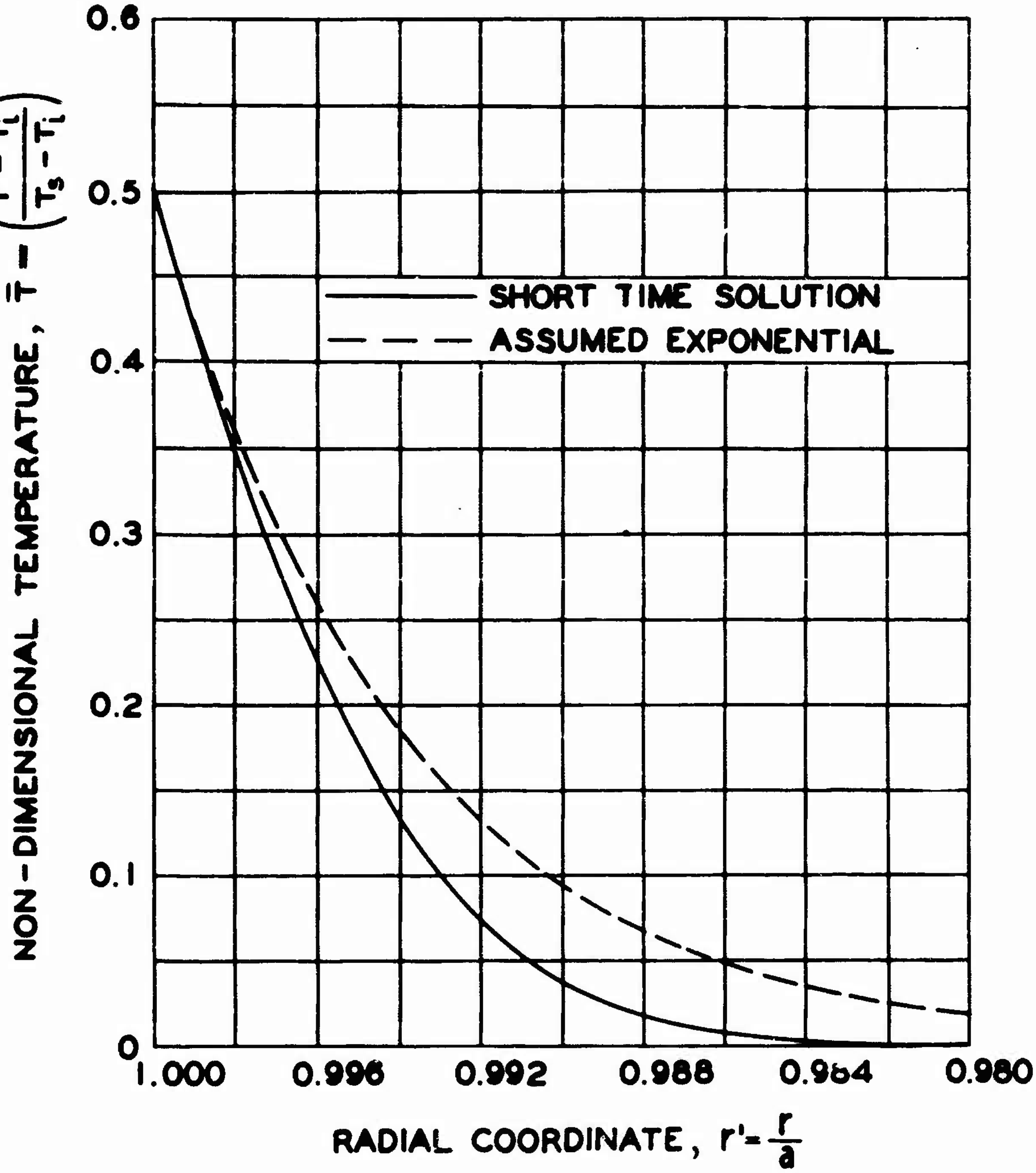

FIG. 2
TEMPERATURE PROFILE AT MELT TIME
$\mu = 165$, $\bar{T}_m = 0.50$

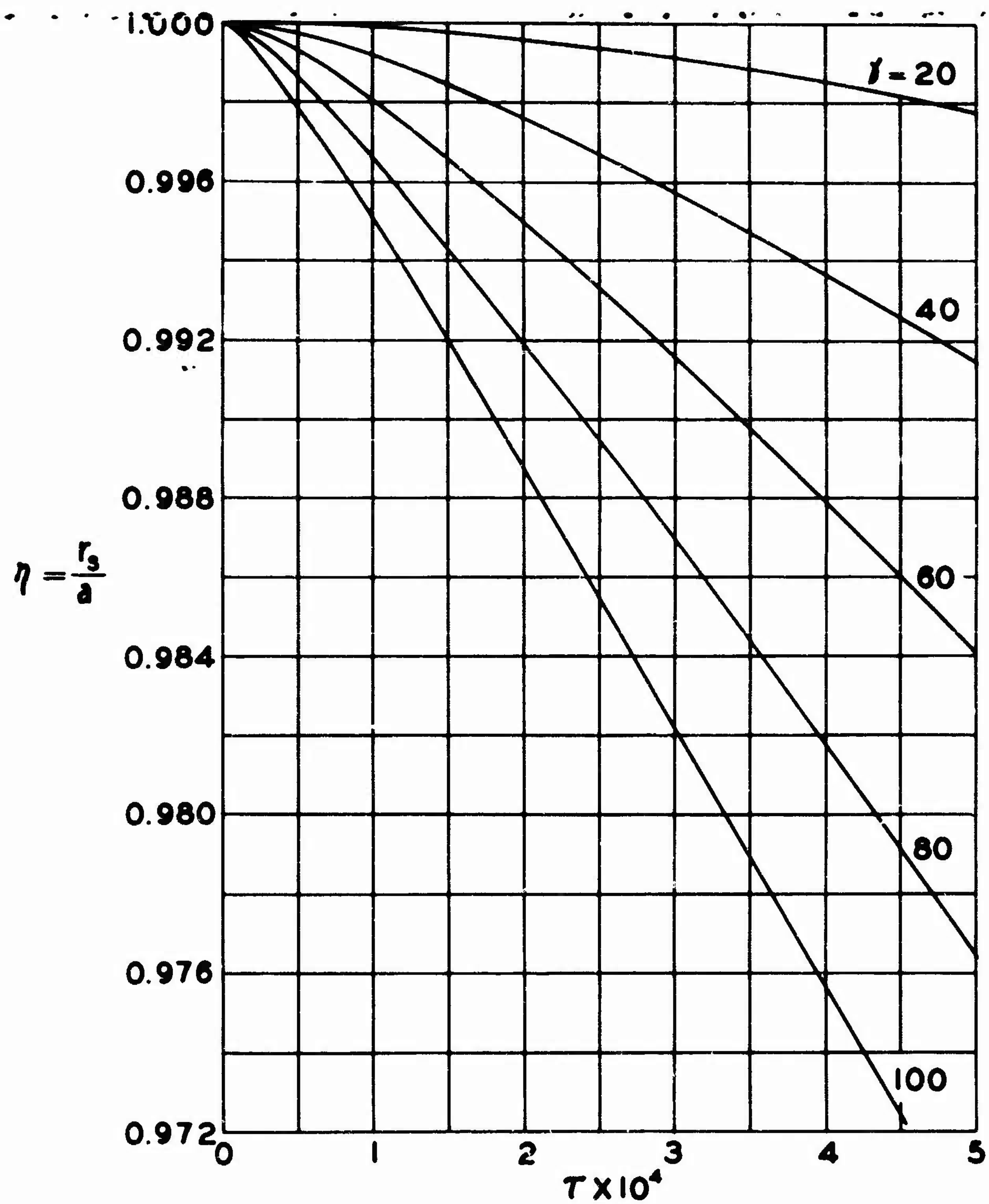

FIG. 3
TIME HISTORY OF ABLATING SURFACE, $\varphi = 0.511$

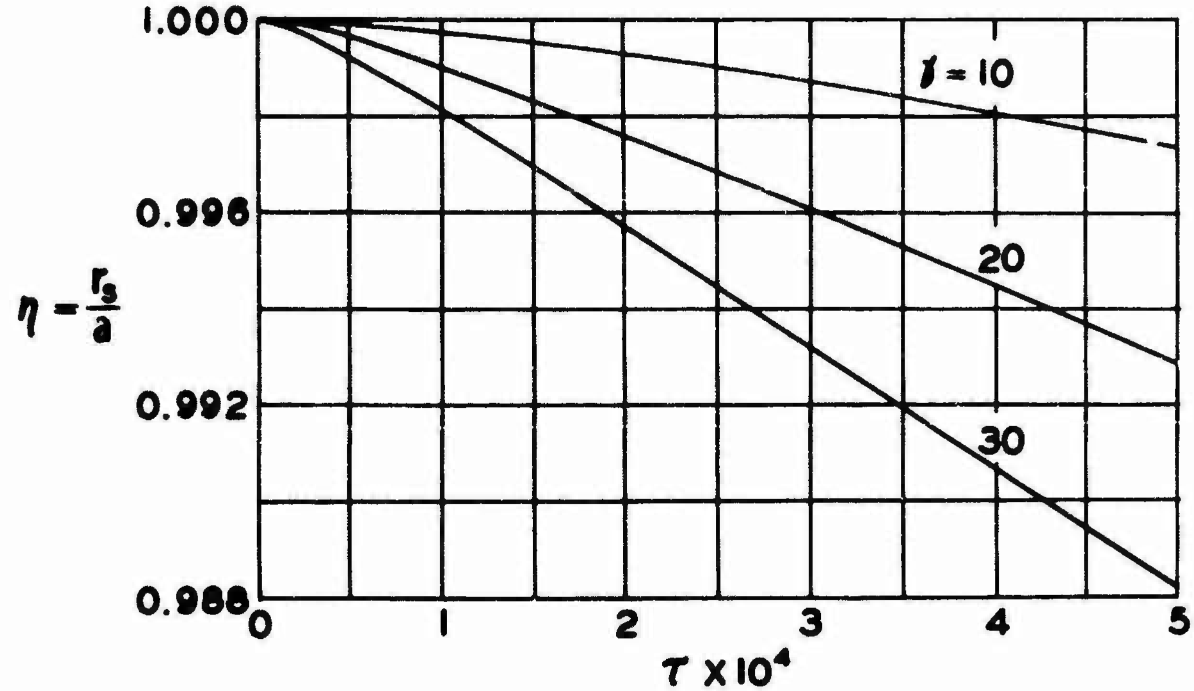

FIG. 4
TIME HISTORY OF ABLATING SURFACE, $\varphi = 0.108$

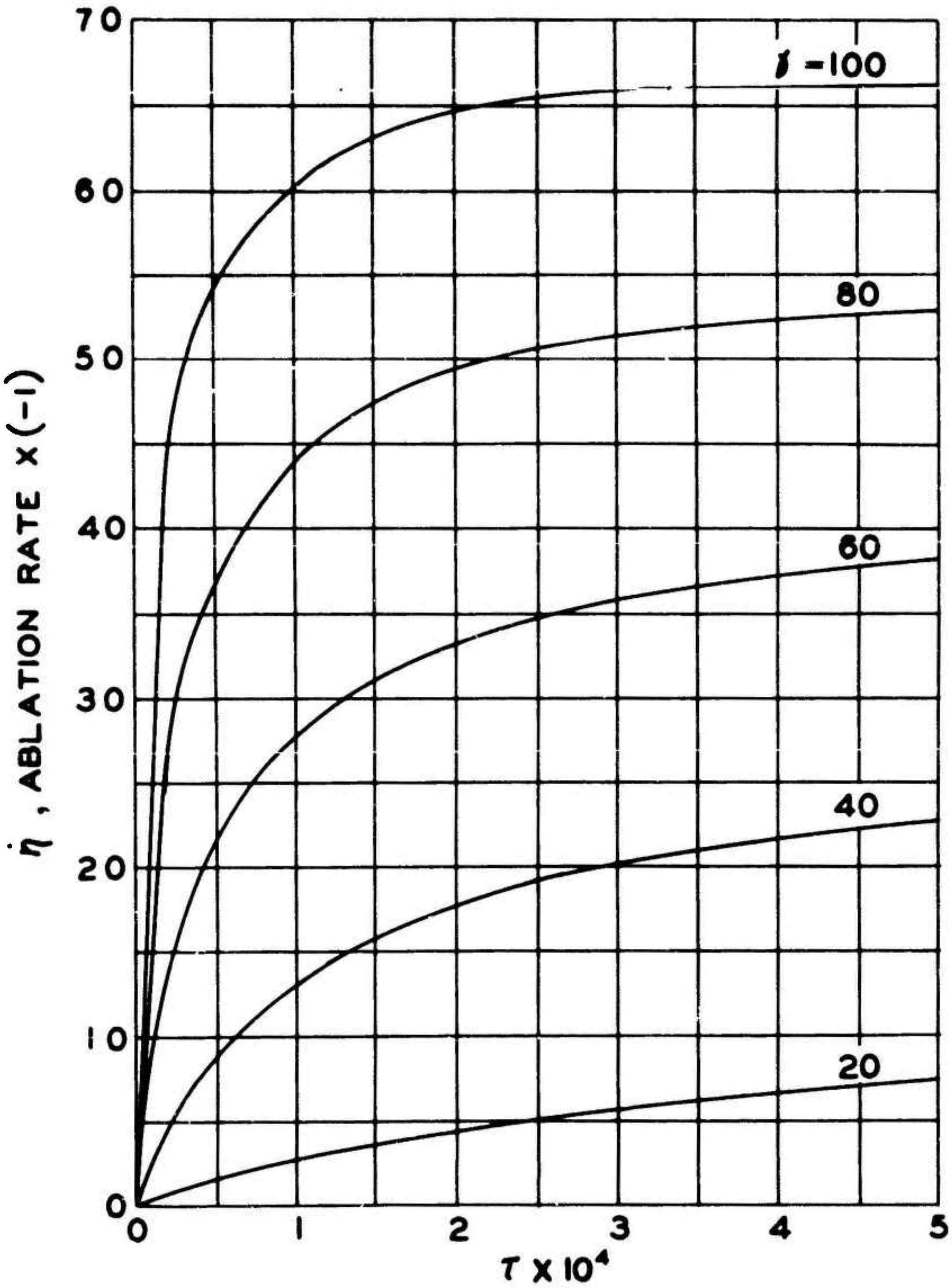

FIG. 5
ABLATION RATE VERSUS TIME, $\varphi = 0.511$

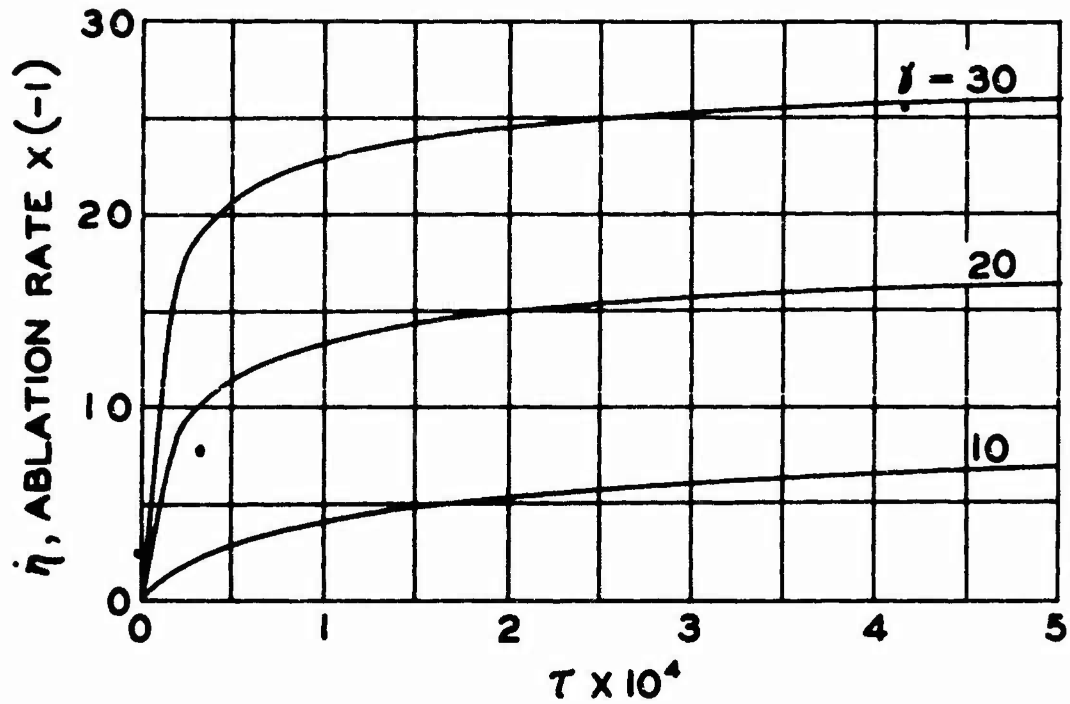

FIG. 6
ABLATION RATE VERSUS TIME, $\varphi = 0.108$